Probabilistic Method/Graph Theory

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Using the Probabilistic Method

The most common way to use the probabilistic method is to have some quantity (a random variable to be specific) X, dependant on some structure, and find $\mathbb{E}X = n$. Then we necessarily have some structure such that $X \geq n$ and another structure such that $X \leq n$.
How do you know when to try to apply the method? Well, the following are some guidelines:

- Proving the existance of something without needing an explicit example
- Situations similar to finding a lower bound for Ramsey's numbers. Key Properties include:
  1. How do you describe an explicit colouring? A complicated colouring would be difficult to describe, let alone checking that no monochromatic K_n exists.
  2. Up to a point, most colourings work, so the method works when expected number of monochromatic K_n's is small
  3. Funky bounds. For example, the bound in question 2 of Probabilistic Method Questions.
- In many situations you won't get the actual bounds (i.e. Ramsey numbers), but typically the bounds are quite good and much better than any explicit methods. So if you are proving an equality or something similar, the method might not be strong enough

General Graph Theory Methods

- Consider extremes, for example the path/cycle/etc. of longest length (in the case of cycle, you need to prove that a cycle exists)
- **Hall's Marriage Theorem**: Let H be a bipartite graph with vertex classes X and Y. Then there exists a matching from X to Y (for each vertex in X, we can match it with one of its neighbours so that the choices are all distinct) if and only if $|\Gamma(A)| \geq |A|$ for all subsets A of X.
- Induction, where you may remove all vertices separately and average over them, or perhaps remove a special vertex, i.e. smallest degree, furthest from some other point, etc.
- Creativitiy. For example, you can prove $R(4,4) > 17$ ($R(4,4) = 18$ in fact) by colouring the edge ij blue if $i - j$ is a quadratic residue mod 17, and yellow otherwise (the graph has vertices $\{1, 2, \ldots, 17\}$).

Probabilistic Method Questions

1. In a tennis tournament with n people, each pair plays each other once. It is called k-fair if for any set of k people, someone has beaten all of them. Prove for all positive integers k there exists an n and some tournament with n people that is k-fair.

2. n people from Canada and n people from Thailand compete in a tournament. Each game is between one person from Canada, and one person from Thailand. If after the competition, it turns out that in any set of 4 people containing two from each country, some pair from different countries did not play each other, call the tournament "cool". Prove that there exists a cool tournament where at least $\frac{3}{4}\left(\frac{n}{\sqrt[3]{n-1}}\right)^2$ games were played.

3. Let G be a graph with n vertices. Show that there exists a set of $\left\lceil \sum_{v \in G} \frac{1}{d(v)+1} \right\rceil$ vertices such that no two are adjacent.

4.(USA) Suppose a, b, c are positive real numbers such that for every positive integer n, $\lfloor an \rfloor + \lfloor bn \rfloor = \lfloor cn \rfloor$. Prove that at least one of a, b, c is an integer.

5. (USAMO 2010) A blackboard contains 68 pairs of nonzero integers. Suppose that for each positive integer k, at most one of (k, k) and $(-k, -k)$ is written. A student erases some of the 136 integers, subject to the condition that no two erased integers may add to 0. The student then scores one point for each of the 68 pairs in which at least one integer is erased. Determine, with proof, the largest number N of points that the student can guarentee to score regardless of which 68 pairs have been written on the board.

General Graph Theory Questions

1. Let G be a graph with $n \geq 2$ vertices, such that $d(v) \geq \frac{n}{2}$ for all vertices v. Show that there exists a Hamiltonian cycle, i.e. a path $v_1 v_2 \dots v_n v_1$ where the v_i are the n distinct vertices of G, and v_i is a neighbour to v_{i+1} for all i.

2. a) Let $k \geq 2$ be a positive integer. Colour the edges of an infinite complete graph (i.e. vertices $v_1, v_2, \dots$ and all possible edges) with k colours. Prove that there exists an increasing sequence of positive integers $n_1, n_2, \dots$ such that all the edges $v_{n_i} v_{n_j}$ for $i \neq j$ have the same colour (i.e. a complete subgraph whose edges all have the same colour).
b) Prove there is an infinite set S of positive integers such that the sum of any two distinct elements of S has an even number of distinct prime factors.

3. There is a dinner for n mathematicians and n physicists. Each person knows strictly more than half of the people of the opposite occupation. Prove that you can seat everyone for dinner around a round table so that: i) Each mathematician is next to two physicists and vice versa
ii) Each person is friends with both of their neighbours.

4. There are $2n$ people at a party where each person has an even number of friends at the party. Prove there are 2 people who have an even number of mutual friends at the party.

5. (IMO 2005) In a mathematics competition with 6 problems, every two of the problems was solved by more than $\frac{2}{5}$ of the contestants. Moreover, no contestant solved all 6 problems. Show that there are at least 2 contestants who solved 5 problems each.